Lagrange's Mean Value Theorem:

21/10/24

Let $f(x)$ be a function such that

i) $f(x)$ is continious on $[a,b]$

ii) $f(x)$ is differentiable on (a,b)

~~f(x) Sc~~ Then $\exists$ atleast a point $c \in (a,b) \ni$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Q) Prove that $\frac{\pi}{6} + \frac{1}{5\sqrt{3}} < \sin^{-1}\left(\frac{3}{5}\right) < \frac{\pi}{6} + \frac{1}{8}$

Sol:- $f(x) = \sin^{-1}x$ in (a,b)

$f'(x) = \frac{1}{\sqrt{1-x^2}}$ in $[a,b]$

$f(x)$ is continious & differentiable on $[a,b]$ & (a,b) respectively.

Then $\exists$ a point $c \in (a,b) \ni$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\frac{1}{\sqrt{1-c^2}} = \frac{\sin^{-1}(b)-\sin^{-1}(a)}{b-a}$$

Since $c \in (a,b)$

$$a < c < b$$

$$a^2 < c^2 < b^2$$

~~1-a~~

$$-a^2 > -c^2 > -b^2$$

$$1-a^2 > 1-c^2 > 1-b^2$$

$$\sqrt{1-a^2} > \sqrt{1-c^2} > \sqrt{1-b^2}$$

$$\frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}}$$

$$\frac{1}{\sqrt{1-a^2}} < \frac{\sin^{-1}(b)-\sin^{-1}(a)}{b-a} < \frac{1}{\sqrt{1-b^2}}$$

$$\frac{b-a}{\sqrt{1-a^2}} < \sin^{-1}(b)-\sin^{-1}(a) < \frac{b-a}{\sqrt{1-b^2}}$$

$a=\frac{1}{2}$ $b=\frac{3}{5}$

$$\frac{\frac{3}{5}-\frac{1}{2}}{\sqrt{1-\frac{1}{4}}} < \sin^{-1}\left(\frac{3}{5}\right)-\sin^{-1}\left(\frac{1}{2}\right) < \frac{\frac{3}{5}-\frac{1}{2}}{\sqrt{1-\frac{9}{25}}}$$

$$\frac{\sqrt{3}}{15} < \sin^{-1}\left(\frac{3}{5}\right)-\cancel{\sin^{-1}}\left(\cancel{\sin}\frac{\pi}{6}\right) < \frac{1}{8}$$

$$\frac{\pi}{6}+\frac{1}{5\sqrt{3}} < \sin^{-1}\left(\frac{3}{5}\right) < \frac{\pi}{6}+\frac{1}{8}.$$

(2) prove that $\frac{\pi}{3}-\frac{1}{5\sqrt{3}} > \cos^{-1}\left(\frac{3}{5}\right) > \frac{\pi}{3}-\frac{1}{8}$

Sol:- $f(x)=\cos^{-1}x$ in $[a,b]$

$f'(x)=\frac{-1}{\sqrt{1-x^2}}$ on (a,b)

$f(x)$ is continious and differentiable on $[a,b]$ & (a,b) respectively

Then $\exists$ a point $c\in(a,b) \ni$

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$-\frac{1}{\sqrt{1-c^2}} = \frac{\cos^{-1}(b)-\cos^{-1}(a)}{b-a}$$

since $c\in(a,b)$

$$a<c<b$$

$$a^2<c^2<b^2$$

$$-a^2>-c^2>-b^2$$

$$1-a^2>1-c^2>1-b^2$$

$$\sqrt{1-a^2}>\sqrt{1-c^2}>\sqrt{1-b^2}$$

$$\frac{1}{\sqrt{1-a^2}}<\frac{1}{\sqrt{1-c^2}}<\frac{1}{\sqrt{1-b^2}}$$

$$-\frac{1}{\sqrt{1-a^2}}>\frac{-1}{\sqrt{1-c^2}}>-\frac{1}{\sqrt{1-b^2}}$$

$$-\frac{1}{\sqrt{1-a^2}} > \frac{\cos^{-1}(b)-\cos^{-1}(a)}{b-a} > -\frac{1}{\sqrt{1-b^2}}$$

$$-\frac{b-a}{\sqrt{1-a^2}} > \cos^{-1}(b)-\cos^{-1}(a) > -\frac{b-a}{\sqrt{1-b^2}}$$

$a=\frac{1}{2}$ $b=\frac{3}{5}$

$$-\frac{\frac{3}{5}-\frac{1}{2}}{\sqrt{1-\frac{1}{4}}} > \cos^{-1}\left(\frac{3}{5}\right)-\cos^{-1}\left(\frac{1}{2}\right) > -\frac{\frac{3}{5}-\frac{1}{2}}{\sqrt{1-\frac{9}{25}}}$$

$$-\frac{\sqrt{3}}{15} > \cos^{-1}\left(\frac{3}{5}\right)-\frac{\pi}{3} > -\frac{1}{8}$$

$$\frac{\pi}{3}-\frac{1}{5\sqrt{3}} > \cos^{-1}\left(\frac{3}{5}\right) > \frac{\pi}{3}-\frac{1}{8}.$$

Q) verify LMVT for $f(x)=\log_e x$ in $[1,e]$

Sol:- i) $f(x)$ is continious on $[1,e]$

ii) $f'(x)=\frac{1}{x}$ So excist on $(1,e)$

$f(x)$ satisfies all condition of LMVT

Then $\exists$ atleast a point ~~c∈(a,b)~~ $c\in(1,e)\ni$

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$\cancel{\frac{1}{c}=\frac{\log b-\log a}{b-a}}$$

$$c=\frac{b-a}{\cancel{\log b-\log a}}$$

$$\frac{1}{c}=\frac{f(e)-f(1)}{e-1}$$

$$\frac{1}{c}=\frac{\log e-\log 1}{e-1}$$

$$\frac{1}{c}=\frac{\log e-0}{e-1}$$

$$\boxed{c=\frac{e-1}{\log e}=e-1}$$ $\log_e e=1$

$c=e-1$

Hence LMVT verified.

Scanned with CamScanner

4) Find 'c' of LMVT for $f(x)=(x-1)(x-2)(x-3)$ on $[0,4]$

Sol:- i) $f(x)$ is continious on $[0,4]$

ii) $f(x)=(x^2-2x-x+2)(x-3)$

$$=(x^2-3x+2)(x-3)=x^3-3x^2-3x^2+9x+2x-6$$

$$f(x)=\cancel{x^3-6x^2+9x}\ x^3-6x^2+11x-6$$

$$f'(x)=3x^2-12x+11$$

So exsist on $(0,4)$

$f(x)$ satisfies all condn of LMVT

Then $\exists$ atleast a point $c\in(0,4)\ni$

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$3c^2-12c+11=\frac{f(4)-f(0)}{4-0}$$

$$\cancel{3c^2-12c+11=\frac{4^3-6(4)^2+11(4)-6+6}{4-}}$$

$$\cancel{3c^2-12c+11-\frac{3}{2}=0}$$

$$\cancel{6c^2-24c+22-3=0}$$

$$\cancel{6c^2-24c+19=0}$$

$$\cancel{c=\frac{12+\sqrt{30}}{6}\quad c=\frac{12-\sqrt{30}}{6}\in(0,4)}$$

~~Hence LMVT verified.~~

$$3c^2-12c+11=3$$

$$3c^2-12c+11-3=0$$

$$3c^2-12c+8=0$$

$$c=\frac{6+2\sqrt{3}}{3}\quad c=\frac{6-2\sqrt{3}}{3}$$

$$c=\frac{6\pm2\sqrt{3}}{3}\in(0,4)$$

Hence LMVT verified.

$\frac{3}{2}-4$

$\frac{3-8}{2}=$